1AA3 Review Session

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Outline

1. Geometric Series
2. Integral Test
3. Remainder Estimate for the Integral Test
4. Partial Sum
5. Comparison Test
6. Absolutely Convergent and Conditionally Convergent
7. Radius of Convergence
8. Other Approaches

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Geometric Series

- The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

Example 1.1

11. Find the sum of the following series

$$\sum_{n=1}^{\infty} \frac{\sin^n x}{3^n}$$

(a) $\dfrac{\sin x - 3}{3}$ **(b)** $\dfrac{3}{\sin x}$ **(c)** $\dfrac{\sin x}{3}$ **(d)** $\dfrac{\sin x}{3 - \sin x}$ **(e)** $\dfrac{3}{3 - \sin x}$

Example 1.2

12. Find the values of x for which the series converges

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{3^n}$$

(a) $(-\frac{2}{3}, \frac{5}{3})$ **(b)** $(-8, -2)$ **(c)** $(-3, 3)$ **(d)** $(-\frac{2}{5}, \frac{2}{5})$ **(e)** $(-1, -4)$

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Integral Test

- Let $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)dx$ is convergent. On the contrary, if $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 2.1

4. Find the values of p for which the series is convergent.

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{p-1}}{n}$$

(a) $p \geq 1$ **(b)** $p < 0$ **(c)** $p \leq 0$ **(d)** $p < 1$ **(e)** $p \leq 1$

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Remainder Estimate for the Integral Test

- Suppose $f(k) = a_k$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx.$$

Example 3.1

3. What is the minimum number of terms needed in order to estimate the sum

$$\sum_{n=1}^{\infty} \frac{1}{(3n+5)^4}$$

correct to within .001?

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** 5

Example 3.2

5. If we use the partial sum s_{10} to approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

estimate the error involved in the approximation.

(a) $\frac{1}{3000}$ **(b)** $\frac{1}{5000}$ **(c)** $\frac{1}{10000}$ **(d)** $\frac{1}{1000}$ **(e)** $\frac{1}{14641}$

Example 3.3

14. What is the minimum number of terms needed in order to estimate the following sum to within .001?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!}$$

(a) 2 **(b)** 3 **(c)** 4 **(d)** 5 **(e)** 6

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Partial Sum

- When we know the partial sum s_n, then we can directly get $a_n = s_n - s_{n-1}$ when $n \geq 2$ and $a_1 = s_1$.

Example 4.1

12. If the n^{th} partial sum of the series $\sum_{n=0}^{\infty} a_n$ is $s_n = \frac{2n+1}{4n+3} - \frac{n}{\ln n}$, find $\sum_{n=0}^{\infty} a_n$.

(a) $\frac{1}{2}$ **(b)** $\frac{1}{3}$ **(c)** $\frac{1}{4}$ **(d)** $\frac{2}{3}$ **(e)** divergent

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Comparison Test

- Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

Example 5.1

7. Using the comparison theorem, which of the following integrals is convergent?

(i) $\displaystyle\int_1^\infty \frac{x\sin^2 x}{\sqrt[3]{1+x^7}}dx$ (ii) $\displaystyle\int_1^\infty \frac{dx}{x+e^{2x}}$ (iii) $\displaystyle\int_2^\infty \frac{x^2}{\sqrt{x^6-1}}dx$

(a) (i) only **(b)** (ii) only **(c)** (i) and (ii) only **(d)** (i) and (iii) only **(e)** (ii) and (iii) only

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Absolutely Convergent and Conditionally Convergent

- For Alternating Series: If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - ...$ where $b_n > 0$. This series satisfies $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} = 0$ then the series is convergent.
- We define a absolutely convergent series $\sum a_n$ if the series of the absolute value $\sum |a_n|$ is convergent.
- A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

Example 6.1

16. Which of the following series are conditionally convergent?

(i) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$ (ii) $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{3^n+2}$

(a) (i) only **(b)** (ii) only **(c)** (i) and (ii) **(d)** neither

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Radius of Convergence

- For a given series $\sum^{\infty} c_n(x-a)^n$, there is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Example 7.1

19. Find the radius of convergence of

$$\sum_{n=1}^{\infty}(n+1)!(3x-1)^n$$

(a) ∞ **(b)** 0 **(c)** 1 **(d)** $\frac{1}{3}$ **(e)** $\frac{2}{3}$

Example 7.2

20. Find the radius of convergence of the following power series.

$$\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)}{n!3^n}x^{3n}$$

(a) $\sqrt[3]{3}$ **(b)** $\frac{1}{\sqrt[3]{3}}$ **(c)** 1 **(d)** 0 **(e)** ∞

Example 7.3

20. Find the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{(2x+3)^n}{2^n(n+1)}$$

(a) 1 **(b)** 2 **(c)** 3 **(d)** $\frac{2}{3}$ **(e)** $\frac{3}{2}$

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The Radio Test

- If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L > 1$ or $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, then no conclusion.

The Root Test

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then no conclusion.

Example 8.1

9. Determine whether the following sequences are convergent or divergent. When convergent, find the limit.

(i) $a_n = \dfrac{(-1)^n n^3}{n^3 + 2n^2 + 1}$

(ii) $a_n = n\sin(n\pi)$

(a) diverges, diverges **(b)** diverges, 0 **(c)** 0, 0 **(d)** 1, diverges **(e)** 1, 0

Example 8.2

9. Determine whether the following sequences are convergent or divergent. When convergent, find the limit.
(i) $a_n = \ln(n+1) - \ln(2n)$
(ii) $a_n = n\sin(1/n)$
(a) divergent, divergent **(b)** divergent, 0 **(c)** $\ln\frac{1}{2}$,divergent **(d)** $\ln\frac{1}{2}, 0$
(e) $\ln\frac{1}{2}$, 1

Yes You Can!